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THREE-DIMENSIONAL DISPLACEMENT DISCONTINUITY SOLUTIONS FOR FLUID-SATURATED POROUS MEDIA

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Abstract—Three-dimensional displacement discontinuity solutions for linear elastic, homogeneous, isotropic, fluid-saturated media are derived. The solutions are obtained from a combination of point force dipoles and an instantaneous fluid source. These solutions are the Green's functions necessary for a numerical boundary element technique known as displacement discontinuity method. © 1998 Elsevier Science Ltd. All rights reserved.

NOMENCLATURE

u_i	displacement $[L]$
σ_{ij}	total stresses $[ML^{-1}T^{-2}]$
p	pore pressure $[ML^{-1}T^{-2}]$
v_i	fluid flux (volume of fluid per unit time per unit area) $[LT^{-1}]$
k	permeability $[LT^{-1}]$
t	real time $[T]$
α	Biot's poroelastic constant
δ_{ij}	Kronecker delta
γ_f	unit weight of fluid $[ML^{-1}T^{-2}]$
κ	diffusivity $[L^2T^{-1}]$
μ	shear modulus $[ML^{-1}T^{-2}]$
v	drained Poisson's ratio
v_u	undrained Poisson's ratio
χ	magnitude of the fluid source $[ML^2T^{-2}]$

INTRODUCTION

Fracture propagation and orientation in fluid-saturated media are controlled by the effective stresses (Cornet and Fairhurst, 1974). The effective stress field, in turn, is dependent on the pore fluid pressure distribution both in space and in time. The rate of change in the stress field is a function of the diffusivity of the medium, and thus, it is partly controlled by the hydraulic conductivity. For fracture problems involving a pore pressure, it is essential that the influence of the pore fluid pressure be determined (Cornet and Fairhust, 1974).

Communication between fracturing fluid and reservoir fluids results in a pore pressure build-up in the immediate surroundings of the fracture. This causes fluid losses and local expansion of the formation. Knowledge of injection rate and pumping time alone is not sufficient to permit the prediction of the extent of a fracture (Geertsma, 1966).

Large shallow earthquakes can induce changes in the pore pressure that are comparable to stress drops on faults. The subsequent re-distribution of pore pressure as a result of fluid flow may result in delayed fracture (Nur and Booker, 1972). This is an attractive mechanism for aftershocks.

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The phenomena described above are governed by the theory of poroelasticity and the field equations were first derived correctly by Biot (1941). Analytical solutions for problems in this type of media are very few and exist only for very simple geometries. Therefore, the use of numerical techniques is mandatory. Fracture propagation and rupture phenomena are numerically modelled best with a displacement discontinuity method (Curran *et al.*, 1985; Crouch and Starfield, 1983). This method is an indirect boundary element method in which a fracture is discretized into line segments whose nodal magnitudes of their aperture (normal DD's) and slip (shear DD's) are the unknowns of the problem. As a fracture propagates, new elements are added to the problem without disturbing the existing discretization, thus, the advantage over domain methods which require re-meshing.

The influences of the aperture and slip of an element on another are obtained by integrating the product of the fundamental solutions (unit point aperture and unit point slip) by their assumed variation over the influencing element. Therefore, these fundamental solutions are the key to the application of the method and are derived in this paper using solutions for the continuous (in time) point force and the instantaneous point fluid source.

FUNDAMENTAL SOLUTIONS

The point force solutions in three-dimensional poroelastic space have been derived by Cleary (1977) and later corrected by Rudnicki (1981, 1987). Rather than deriving the displacement discontinuity solutions from the governing equations (equilibrium, compatibility and diffusion), we can use the point force solutions and the fluid source solutions as the starting point. It has been shown by Wiles and Curran (1982) that there exists a relationship between point force dipoles and displacement discontinuities in elastic media. This relationship is described later in the paper. Following the same line of thought, Curran and Carvalho (1987) have derived the relationship between point force dipoles, fluid sources and displacement discontinuities in two-dimensional poroelastic media. These relationships can be extended to three-dimensions and are as follows. The normal displacement discontinuity is obtained by applying to the medium three sets of double forces (dipoles). The application of the sets of double forces initiates a diffusion process which causes the magnitude of the displacement discontinuity to change with time. In order to maintain the magnitude constant in time (continuous DD), an instantaneous point fluid source is applied simultaneously with the double forces. To obtain a unidirectional displacement discontinuity two of the three sets of double forces are adjusted to prevent lateral movement at the point of application. The unidirectionality of the normal displacement discontinuity (point aperture) is very important in the development of the numerical technique because it properly isolates the relative fracture motion in three orthogonal directions. The shear displacement discontinuities are obtained by applying to the medium, two sets of double forces with moment. This creates a state of pure shear, causing no change in volume, and therefore, no diffusion process, at the point of application. These relationships are shown in Fig. 1 and are established in the next sections.

INSTANTANEOUS DISPLACEMENT DISCONTINUITY SOLUTIONS

The solutions for instantaneous displacement discontinuities can be obtained from their continuous counterparts by differentiation with respect to time. It should be noted that the solutions contain an instantaneous part which is identical to the elastic solutions written in terms of the undrained parameters and an evolving part which is time dependent (Detournay and Cheng, 1987).

EQUIVALENCES BETWEEN POINT FORCES, FLUID SOURCES AND DISPLACEMENT DISCONTINUITIES

Wiles and Curran (1982) derived the solutions for a singular displacement discontinuity (Green's function) for a homogeneous, linear elastic, isotropic continuum (pure elasticity) by solving the equilibrium equations subjected to the following boundary condition:

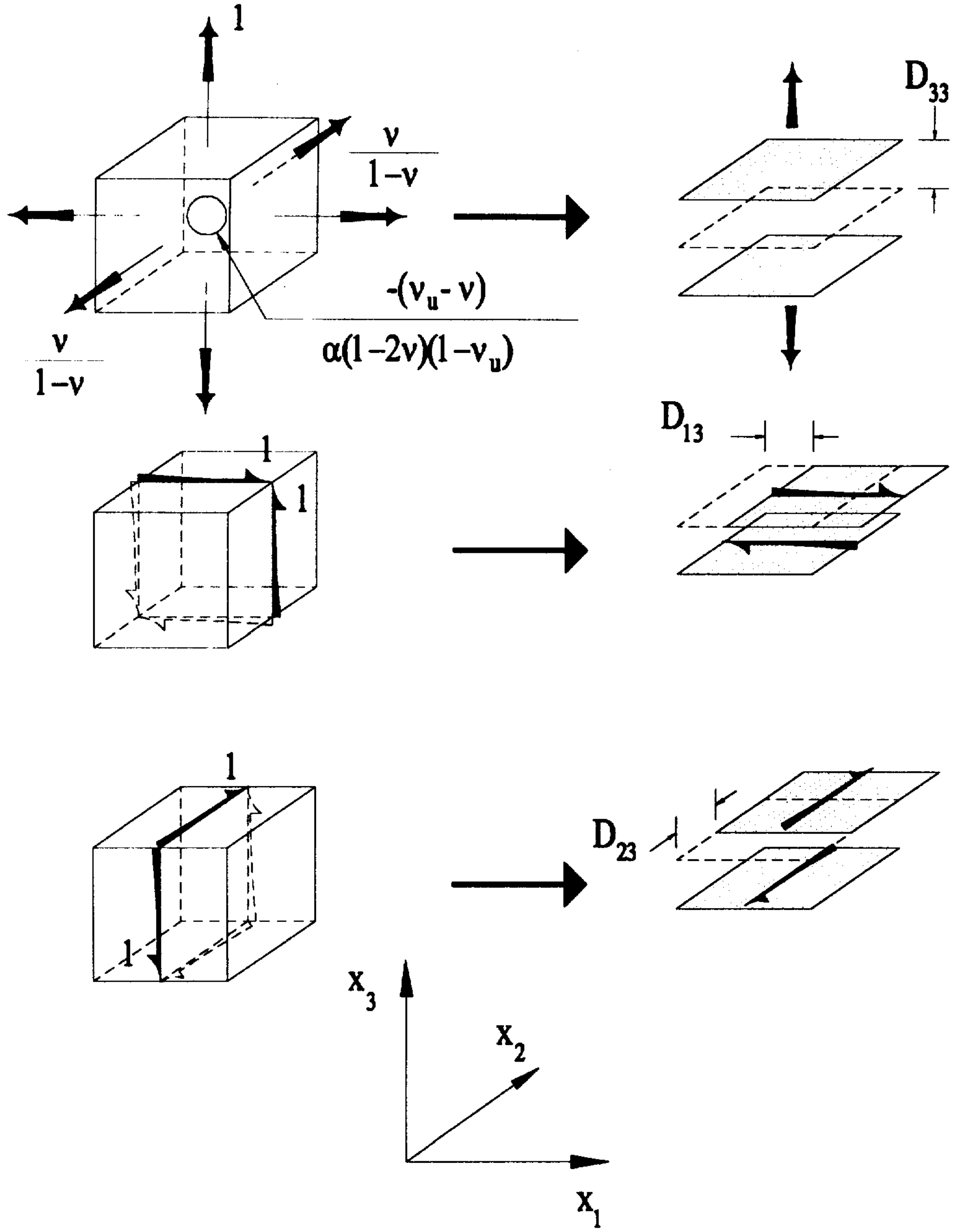

Fig. 1. Equivalence between point forces, fluid sources and displacement discontinuities (3-D).

$$Lu_i(x_1, x_2, x_3) = \delta(x_1, x_2, x_3) \quad i = 1, 2, 3$$

where

$$L = \lim_{x_3 \to 0^+} - \lim_{x_3 \to 0^-}$$

and $\delta(x_1, x_2, x_3)$ is the Dirac delta function.

In addition, the condition that the displacements and stresses vanish are infinity was also satisfied. Upon examination of the solution due to quadrupoles and hexapoles reported by Brady and Bray (1978), they found that these solutions exhibited basically the same behaviour as the displacement discontinuity solutions, differing only by multiplication by a material property, namely:

Unit normal displacement discontinuity (D_{33}):

$$D_{33} \equiv \frac{-2\mu(1-v)}{1-2v}\left\{\frac{\partial P_3}{\partial x_3}+\frac{v}{1-v}\left[\frac{\partial P_1}{\partial x_1}+\frac{\partial P_2}{\partial x_2}\right]\right\} \tag{1}$$

where
$\partial P_3/\partial x_3$ is a unit double force in the x_3-direction
$\partial P_2/\partial x_2$ is a unit double force in the x_2 direction
$\partial P_1/\partial x_1$ is a unit double force in the x_1 direction

Unit shear displacement discontinuity (D_{13}):

$$D_{13} \equiv -\mu\left(\frac{\partial P_1}{\partial x_3}+\frac{\partial P_3}{\partial x_1}\right) \tag{2}$$

where
$\partial P_1/\partial x_3$ is a unit double force in the x_1 direction with moment about the x_3 direction
$\partial P_3/\partial x_1$ is a unit double force in the x_3 direction with moment about the x_1 direction

Unit shear displacement discontinuity (D_{23}):

$$D_{23} \equiv -\mu\left(\frac{\partial P_2}{\partial x_3}+\frac{\partial P_3}{\partial x_2}\right) \tag{3}$$

where
$\partial P_2/\partial x_3$ is a unit double force in the x_2 direction with moment about the x_3 direction
$\partial P_3/\partial x_2$ is a unit double force in the x_3 direction with moment about the x_2 direction

Following this methodology, the solutions for point forces in three-dimensional poroelastic media were used to construct the required displacement jumps at the point of their application. Differentiating the displacement solution due to point forces, we obtain the solution due to double forces and double forces with moment. Denoting the quadrupoles used by Brady and Bray (1978) by

$$\begin{aligned} Q_{s1} &= \frac{\partial P_1}{\partial x_3}+\frac{\partial P_3}{\partial x_1} \\ Q_{s2} &= \frac{\partial P_2}{\partial x_3}+\frac{\partial P_3}{\partial x_2} \\ Q_n &= \frac{\partial P_3}{\partial x_3}+\frac{v}{1-v}\left[\frac{\partial P_1}{\partial x_1}+\frac{\partial P_2}{\partial x_2}\right] \end{aligned} \tag{4}$$

Where P_1, P_2 and P_3 are the point force solutions derived by Cleary (1977), it is found that the displacement jumps generated by these quadrupoles in poroelastic media are given by:

$$\begin{aligned} Lu_3^{Q_{s1}}(x_1,x_2,x_3,t) &= 0 \\ Lu_1^{Q_{s1}}(x_1,x_2,x_3,t) &= -\frac{1}{\mu}\delta(x_1,x_2,x_3)H(t) \\ Lu_2^{Q_{s1}}(x_1,x_2,x_3,t) &= 0 \end{aligned} \tag{5}$$

$$
\begin{aligned}
&Lu_3^{Q_{s2}}(x_1, x_2, x_3, t) = 0 \\
&Lu_1^{Q_{s2}}(x_1, x_2, x_3, t) = 0 \\
&Lu_2^{Q_{s2}}(x_1, x_2, x_3, t) = -\frac{1}{\mu}\delta(x_1, x_2, x_3)H(t)
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
Lu_3^{Q_n}(x_1, x_2, x_3, t) &= \frac{-(1-2v)}{2\mu(1-v)}\delta(x_1, x_2, x_3)H(t) \\
&\quad + \frac{v_u - v}{8\pi(1-v)(1-v_u)} L\left[\frac{x_3}{R_3}\mathrm{erf}(\xi) - \frac{2\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}\right] \\
Lu_1^{Q_n}(x_1, x_2, x_3, t) &= \frac{v_u - v}{8\pi(1-v)(1-v_u)} L\left[\frac{x_1}{R^3}\mathrm{erf}(\xi) - \frac{2\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}\right] \\
Lu_2^{Q_{s2}}(x_1, x_2, x_3, t) &= \frac{v_u - v}{8\pi(1-v)(1-v_u)} L\left[\frac{x_2}{R^3}\mathrm{erf}(\xi) - \frac{2\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}\right]
\end{aligned} \tag{7}
$$

where

$$
\xi = \frac{R}{2\sqrt{\kappa t}}
$$

As eqns (5) and (6) show, the displacement jump generated by the shear quadrupole is the correct one when multiplied by the material constant, $-\mu$. Therefore, the solution for a shear displacement discontinuity in poroelastic media is obtained by the same procedure as that for elastic media. As was mentioned before, this is due to the fact that the shear quadrupole generates a state of pure shear at its point of application and no diffusion process is present.

However, the normal hexapole does not generate the proper jump in the displacement field. In order to obtain a unit constant (in time) jump in the displacement in the x_3 direction and no displacement jump in the x_1 and x_2 directions it is necessary to eliminate the last term on the R.H.S. of eqn (7).

Because these terms have the same form as the displacements in the x_3, x_1 and x_2 directions, respectively, due to an instantaneous fluid source, we simply need to determine the magnitude of the source which will cancel them. This is achieved by equating the solution for the displacements due to the instantaneous fluid source to the negative of the terms to be cancelled, i.e.,

$$
\chi\frac{\alpha(1-2v)}{8\pi\mu(1-v)}\frac{x_i}{R^3}\left(\mathrm{erf}(\xi) - \frac{2\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}\right) = -\frac{v_u - v}{8\pi\mu(1-v)(1-v_u)}\frac{x_i}{R^3}\left(\mathrm{erf}(\xi) - \frac{2\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}\right) \tag{8}
$$

where χ is the magnitude of the fluid source. Solving yields

$$
\chi = -\frac{v_u - v}{\alpha(1-2v)(1-v_u)} \tag{9}
$$

Therefore, the displacements, tractions, pore pressure and fluid fluxes due to a unit normal ($n = k$) or shear ($n \neq k$) DD both in three dimensions are given by

$$u_{ikn}^{dc} = \frac{-\mu(1-\nu)}{1-2\nu}\left\{\frac{1-2\nu}{1-\nu}(u_{ik,n}^{pc}+u_{in,k}^{pc})+2\delta_{kn}\left[\frac{\nu}{1-\nu}u_{il,l}^{pc}-\frac{\nu_u-\nu}{\alpha(1-2\nu)(1-\nu_u)}u_i^{si}\right]\right\}$$

$$u_{ijkn}^{dc} = \frac{-\mu(1-\nu)}{1-2\nu}\left\{\frac{1-2\nu}{1-\nu}(\sigma_{ijk,n}^{pc}+\sigma_{ijn,k}^{pc})+2\delta_{kn}\left[\frac{\nu}{1-\nu}\sigma_{ijl,l}^{pc}-\frac{\nu_u-\nu}{\alpha(1-2\nu)(1-\nu_u)}\sigma_{ij}^{si}\right]\right\}$$

$$p_{kn}^{dc} = \frac{-\mu(1-\nu)}{1-2\nu}\left\{\frac{1-2\nu}{1-\nu}(p_{k,n}^{pc}+p_{n,k}^{pc})+2\delta_{kn}\left[\frac{\nu}{1-\nu}p_{l,l}^{pc}-\frac{\nu_u-\nu}{\alpha(1-2\nu)(1-\nu_u)}p^{si}\right]\right\}$$

$$v_{ikn}^{dc} = \frac{-\mu(1-\nu)}{1-2\nu}\left\{\frac{1-2\nu}{1-\nu}(v_{ik,n}^{pc}+v_{in,k}^{pc})+2\delta_{kn}\left[\frac{\nu}{1-\nu}v_{il,l}^{pc}-\frac{\nu_u-\nu}{\alpha(1-2\nu)(1-\nu_u)}v_i^{si}\right]\right\} \tag{10}$$

where

$i, j, k = 1, 2, 3$

u_{ik}^{pc}, u_i^{si} are the displacements due to a unit continuous point force and an instantaneous fluid source, respectively.

σ_{ijk}^{pc}, σ_{ij}^{si} are the stresses due to a unit continuous point force and an instantaneous fluid source, respectively.

p_k^{pc}, p^{si} are the pore pressures due to a unit continuous point force and an instantaneous fluid source, respectively.

v_{ik}^{pc}, v_i^{si} are the fluxes due to a unit continuous point force and an instantaneous fluid source, respectively.

Repeated subscripts denote the usual summation convention and a comma denotes partial differentiation. Superscripts refer to the type of singular solution. The first superscript denotes the type of physical singularity, i.e., point force (p), displacement discontinuity (d) or fluid source (s), and the second one denotes its temporal form, i.e., instantaneous (i) or continuous (c). The expressions for the displacements, stresses, pore pressures and pore fluid flow due to continuous normal and shear unit displacement discontinuities are given in the Appendix.

The equivalences between point forces, fluid sources and displacement discontinuities result in the following equalities:

$$u_{kni}^{dc} = -\sigma_{ikn}^{pc}$$

$$p_{kn}^{dc} = 2\mu(\delta_{kn}p_{l,l}^{pc}-p_{n,k}^{pc})$$

$$p_i^{pc} = \frac{2\mu(1-\nu)}{1-2\nu}\frac{\nu_u-\nu}{\alpha^2(1-2\nu)(1-\nu_u)}u_i^{si} = \frac{\gamma_f\kappa}{k}u_i^{si}$$

$$p_{ij}^{dc} = -\frac{2\mu(1-\nu)}{1-2\nu}\frac{\nu_u-\nu}{\alpha^2(1-2\nu)(1-\nu_u)}\sigma_{ij}^{si} = -\frac{\gamma_f\kappa}{k}u_{ij}^{si} \tag{11}$$

CONCLUSION

In a poroelastic system it is necessary to measure five independent material properties to describe the response of the medium. The properties chosen to express the solutions were the shear modulus (ν), Biot's poroelastic constant (α), the drained and undrained Poisson's ratios (ν, ν_u) and the permeability (k). This choice allows an immediate check of the asymptotic behaviour of the solutions. At time $t = 0^+$ the system responds as an elastic system with undrained properties. On the other hand, at time $t = \infty$ the pore pressures and fluxes vanish and the system responds in an undrained manner. In addition to this expected behaviour, the displacements and fluxes exhibit a jump at the point of application of the DD. The jump in displacement is permanent and constant in time. The jump in flux has its maximum at the time of application of the DD and fades at large times.

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APPENDIX

$$R^2 = r^2 + x_3^2 = x_1^2 + x_2^2 + x_3^2$$

$$\xi = \frac{R}{2\sqrt{\kappa t}}$$

$\mathrm{erf}(x)$ = error function of x and is defined as

$$\mathrm{erf}(x) = \int_0^x \mathrm{e}^{-u^2}\,\mathrm{d}u$$

$$p_{kn}^{dc} = \frac{\mu}{2\pi(1-2\nu)\alpha}\frac{\nu_u-\nu}{1-\nu_u}\frac{1}{R^3}\left\{\delta_{kn}\left[(2+4\xi^2)\frac{\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}-\mathrm{erf}(\xi)\right]+\frac{x_kx_n}{R^2}\left[3\,\mathrm{erf}(\xi)-(6+4\xi^2)\frac{\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}\right]\right\}$$

$$v_{ikn}^{dc} = \frac{k\mu}{2\pi(1-2\nu)\alpha\gamma_f}\frac{\nu_u-\nu}{1-\nu_u}\frac{1}{R^5}\left\{(\delta_{kn}x_i+\delta_{ik}x_n+\delta_{in}x_k)\left[(6+4\xi^2)\frac{\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}-3\,\mathrm{erf}(\xi)\right]\right.$$
$$\left.+8\delta_{kn}x_i\frac{\xi^5}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}+\frac{x_ix_kx_n}{R^2}\left[15\,\mathrm{erf}(\xi)-(30+20\xi^2+8\xi^4)\frac{\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}\right]\right\}$$

$$u_{ikn}^{dc} = \frac{1}{8\pi(1-\nu)}\frac{1}{R^3}\left\{(1-2\nu)(\delta_{ik}x_n+\delta_{in}x_k-\delta_{kn}x_i)+\frac{3x_ix_kx_n}{R^2}+\frac{\nu_u-\nu}{1-\nu_u}\left\{\delta_{kn}x_i\left[\left(1+\frac{3}{2\xi^2}\right)\mathrm{erf}(\xi)-\left(4+\frac{3}{\xi^2}\right)\frac{\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}\right]\right.\right.$$
$$\left.\left.-(\delta_{ik}x_n+\delta_{in}x_k)\left[\left(1-\frac{3}{2\xi^2}\right)\mathrm{erf}(\xi)+\frac{3}{\xi\sqrt{\pi}}\mathrm{e}^{-\xi^2}\right]+\frac{3x_ix_kx_n}{R^2}\left[\left(1-\frac{5}{2\xi^2}\right)\mathrm{erf}(\xi)+\left(\frac{4}{3}+\frac{5}{\xi^2}\right)\frac{\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}\right]\right\}\right\}$$

$$\sigma_{ijkn}^{dc} = \frac{\mu}{4\pi(1-\nu)}\frac{1}{R^3}\left\{(1-2\nu)(\delta_{in}\delta_{jk}+\delta_{ik}\delta_{jn})-(1-4\nu)\delta_{ij}\delta_{kn}-\frac{15x_ix_jx_kx_n}{R^4}\right.$$
$$+\frac{3}{R^2}[(1-2\nu)(\delta_{ij}x_nx_k+\delta_{kn}x_ix_j)+\nu[(\delta_{ik}x_n+\delta_{in}x_k)x_j+(\delta_{jk}x_n+\delta_{jn}x_k)x_i]]$$
$$+\frac{\nu_u-\nu}{1-\nu_u}\left\{\delta_{ij}\delta_{kn}\left[\left(3+\frac{3}{2\xi^2}\right)\mathrm{erf}(\xi)-\left(8+\frac{3}{\xi^2}+8\xi^2\right)\frac{\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}\right]\right.$$
$$-(\delta_{in}\delta_{jk}+\delta_{ik}\delta_{jn})\left[\frac{3}{\xi\sqrt{\pi}}\mathrm{e}^{-\xi^2}+\left(1-\frac{3}{2\xi^2}\right)\mathrm{erf}(\xi)\right]$$
$$+\frac{3}{R^2}\left[(\delta_{ij}x_nx_k+\delta_{kn}x_ix_j)\left[\left(\frac{16}{3}+\frac{5}{\xi^2}+\frac{8\xi^2}{3}\right)\frac{\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}-\left(1+\frac{5}{2\xi^2}\right)\mathrm{erf}(\xi)\right]\right.$$
$$\left.+[(\delta_{ik}x_n+\delta_{in}x_k)x_j+(\delta_{jk}x_n+\delta_{jn}x_k)x_i]\left[\left(\frac{4}{3}+\frac{5}{\xi^2}\right)\frac{\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}+\left(1-\frac{5}{2\xi^2}\right)\mathrm{erf}(\xi)\right]\right]$$
$$\left.\left.-\frac{15x_ix_jx_kx_n}{R^4}\left[\left(\frac{8}{3}+\frac{7}{\xi^2}+\frac{8\xi^2}{15}\right)\frac{\xi}{\sqrt{\pi}}\mathrm{e}^{-\xi^2}+\left(1-\frac{7}{2\xi^2}\right)\mathrm{erf}(\xi)\right]\right\}\right\}$$